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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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OFFICE ... 9-JAN 1930 MADRAS

EXPRESSIONS FOR THE CLASS NUMBER OF BINARY QUADRATIC FORMS.

BY S. D. CHOWLA.

1. Hacks* has found that the number of properly primitive classes of negative determinant $-q$, where q is a prime of the form $4n+3$ is

$$\frac{q-1}{2} - 2\sum_{s=1}^{\frac{1}{2}(q-3)} (-1)^s \left[\sqrt{\tfrac{1}{2} qs}\right]$$

where $[x]$ denotes the greatest integer contained in x.

In this note it is proved that the expression

$$4\sum_{0}^{\frac{1}{4}(p-5)} \left[\sqrt{sp+\frac{p}{4}}\right] - \tfrac{1}{6}(p-1)(2p-7)$$

gives the class number for the negative determinant $-p$, where p is a prime of the form $4m+1$.

2. Let θ be the number of integers x such that

$$x^2 \equiv \alpha \ (\text{mod. } p),$$

where $$0 < \alpha < \tfrac{1}{4}p, \text{ and } 0 < x < \tfrac{1}{2}p.$$

Obviously

$$\theta = \sum \left(\frac{\alpha}{p}\right), \qquad \ldots \quad (1)$$

where α takes only those values between 0 and $p/4$ for which $(\alpha/p) = +1$.

On the other hand it is clear that

$$\theta = \sum_{s=0}^{\frac{1}{4}(p-5)} \left[\sqrt{sp+\frac{p}{4}}\right] - \sum_{s=1}^{\frac{1}{4}(p-5)} \left[\sqrt{sp}\right]. \qquad \ldots \quad (2)$$

* *Acta Math.*, 14, 1890—1, 321—328.

Again

$$\sum_{s=1}^{\frac{1}{2}(p-1)} s^2 = \sum_{0<a<p} (a) + p \sum_{s=1}^{m} s([\sqrt{(s+1)p}] - [\sqrt{sp}]). \qquad \ldots \quad (3)$$

where a runs through quadratic residues of p, and $m = \left[\frac{(p-1)^2}{4p}\right]$.

From (1), (2) and (3) we easily deduce that :

$$h(-p) = 2 \sum_{0}^{\frac{1}{4}p} \left(\frac{n}{p}\right) = 4 \left\{ \sum_{0}^{\frac{1}{4}(p-5)} \left[\sqrt{sp + \frac{p}{4}}\right] \right\} - \frac{1}{6}(p-1)(2p-7).$$

3. The following related results seem to be of some interest.

$$\sum_{s=1}^{p-1} U_{sp} = 0$$

where p is an odd prime and $U_n = (-1)^{[\sqrt{n}]}$. $\qquad \ldots \qquad \ldots \quad (4)$

$$\sum_{s=1}^{8p^2} \left\{ (-1)^{[\sqrt[3]{sp}]} \right\} = -6p \qquad \ldots \qquad \ldots \quad (5)$$

where $p \equiv 2 \pmod 3$ is a prime greater than 5.

Results similar to (4) have been given by Berger (*Nova Acta Reg Soc. Sc. Upsaliensis*, (3), 11, 1883, No. 7.

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THE EVALUATION OF A TRIGONOMETRICAL SUM.

BY S. D. CHOWLA.

1. It is well-known that if p is a prime of the form $4m + 3$, then

$$\sum_{1}^{\frac{1}{2}(p-1)} \cot \frac{n^2\pi}{p} = A \sqrt{p},$$

where A is an integer whose value depends on the binary quadratic form class-number. The object of this note is to obtain a similar evaluation of the sum

$$\sum \left\{ \cot \left(\frac{n^2\pi}{6p}\right) + \cot \left(\frac{5n^2\pi}{6p}\right) \right\},$$

where n takes all positive integral values less than and prime to $6p$, and p is a prime of the form $20m + 11$.

2. If

$$S(x) = \frac{\sin x}{1} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 11x}{11} + \dots \qquad \dots \quad (1)$$

it is easy to prove that

$$\left.\begin{array}{ll} S(x) = \pi/6, & 0 < x < \pi/3 \;; \\ S(x) = \pi/3, & \pi/3 < x < 2\pi/3 \,; \\ S(x) = \pi/6, & 2\pi 13 < x < \pi. \end{array}\right\} \qquad \dots \quad (2)$$

Also if p is a prime of the form $4m + 3$), we know that

$$\sum_{n=1}^{\frac{1}{2}(p-1)} \left(\frac{n}{p}\right) \sin \frac{2mn\pi}{p} = \tfrac{1}{2} \left(\frac{m}{p}\right) \sqrt{p}, \qquad \dots \quad (3)$$

where $\left(\frac{n}{p}\right)$ is Legendre's quadratic-residue symbol, equal to zero if $n \equiv 0 \pmod{p}$. From (1) and (3) we get

$$\sum_{n=1}^{\frac{1}{2}(p-1)} \left(\frac{n}{p}\right) S\left(\frac{2n\pi}{p}\right) = \tfrac{1}{2} \sqrt{p} \sum \frac{1}{m} \left(\frac{m}{p}\right), \qquad \dots \quad (4)$$

where m takes all positive integral values prime to $6p$.

Since

$$\frac{1}{a}-\frac{1}{b-a}+\frac{1}{b+a}-\frac{1}{2b-a}+\frac{1}{2b+a}-\quad\ldots\quad=\frac{\pi}{b}\cot\left(\frac{\pi a}{b}\right),$$

the right-hand side of (4) is equal to

$$\frac{1}{2}\sqrt{p}\cdot\frac{\pi}{6p}\cdot\sum_{\left\{\substack{3p>m>0\\(m,\,6p)=1}\right\}}\left(\frac{m}{p}\right)\cot\frac{m\pi}{6p}\,. \qquad \ldots \quad (5)$$

From (2) we see that if p is a prime of the form $4m+3$ $(m>0)$i then the left-hand side of (4) is equal to

$$\frac{\pi}{6}\left\{\sum_{1}^{\frac{1}{6}p}\left(\frac{n}{p}\right)+2\sum_{\frac{1}{6}p+1}^{\frac{1}{3}p}\left(\frac{n}{p}\right)+\sum_{\frac{1}{3}p+1}^{\frac{1}{2}p}\left(\frac{n}{p}\right)\right\}. \qquad \ldots \quad (6)$$

From (4), (5) and (6) we get

$$\sum\left(\frac{m}{p}\right)\cot\frac{m\pi}{6p}=2\sqrt{p}\left\{\sum_{1}^{\frac{1}{6}p}\left(\frac{n}{p}\right)+2\sum_{\frac{1}{6}p+1}^{\frac{1}{3}p}\left(\frac{n}{p}\right)+\sum_{\frac{1}{3}p+1}^{\frac{1}{2}p}\left(\frac{n}{p}\right)\right\} \qquad \ldots \quad (7)$$

where m takes values between 0 and $3p$ which are prime to $6p$, and $p\equiv 3$ (mod. 4) is a prime greater than 3. From (7) it is easy to deduce that if p is a prime $\equiv 11$ or 19 (mod. 20), then

$$\sum_{n}\left\{\cot\left(\frac{n^2\pi}{6p}\right)+\cot\left(\frac{5n^2\pi}{6p}\right)\right\}$$

$$=8\sqrt{p}\left\{\sum_{1}^{\frac{1}{6}p}\left(\frac{n}{p}\right)+2\sum_{1+\frac{1}{6}p}^{\frac{1}{3}p}\left(\frac{n}{p}\right)+\sum_{1+\frac{1}{3}p}^{\frac{1}{2}p}\left(\frac{n}{p}\right)\right\}, \qquad \ldots \quad (8)$$

where, on the left, n takes all positive integral values less than and prime to $6p$.

THE BENDING OF A THIN ELLIPTIC PLATE WITH SUPPORTED EDGE UNDER UNIFORM NORMAL PRESSURE

By B. B. Sen, Islamia College, Calcutta.

1. The problem of the bending of a thin elliptic plate with supported rim subjected to uniform normal pressure was considered by B. Galerkin in a paper named * "Adaptation of curvilinear isothermal co-ordinates to integrate the equations of equilibrium of elastic plates." The deflexion ω obtained by him satisfies the conditions that ω and the normal pressures vanish at the rim but does not strictly make the bending moment about a tangent line of the rim zero.

In this paper a much simpler value for the deflexion has been obtained which becomes zero at the supported edge, and also makes the bending moment about a tangent line of the rim vanish. No general solution has been assumed, but the equation of equilibrium has been integrated step by step with the introduction of suitable constants.

2. If Z be the load per unit area and D the flexural rigidity of the plate, then the transverse displacement ω is given by the equation †

$$\nabla_1^4 \omega = \frac{Z}{D}. \qquad \ldots \quad (1)$$

At the supported edge ω and G vanish, where

$$G = -D \left[\frac{\partial^2\omega}{\partial \nu^2} + \sigma \left(\frac{\partial^2\omega}{\partial s^2} + \frac{1}{\rho}\ \frac{\partial\omega}{\partial \nu}\right) \right]. \qquad \ldots \quad (2)$$

Here s denotes the direction of the tangent to the bounding curve, ν the direction of the normal, ρ the radius of curvature of the bounding curve and σ the Poisson's ratio.

3. If we take

$$x + iy = c \cosh (\xi + i\eta),$$

we know that ξ = constant = α (say) represents an ellipse.

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* *Messenger of Mathematics*, Vol. 52 (1923).

† See Love's *Elasticity*, (4th edition), p. 488.

Then
$$x = c \cosh \xi \cos \eta$$
$$y = c \sinh \xi \sin \eta$$

and $$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = \frac{c^2}{2}(\cosh 2\xi - \cos 2\eta) \quad \ldots \quad (3)$$

The equation $\nabla_1{}^4 \omega = Z/D$ can now be written in the form

$$\left[\frac{1}{\cosh 2\xi - \cos 2\eta}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\right]^2 \omega = \frac{c^4 Z}{4D} = 8\, p_0 \text{ (say)} \ldots \quad (4)$$

Since
$$\frac{\partial \omega}{\partial \nu} = h \frac{\partial \omega}{\partial \xi} \text{ and } \frac{\partial \omega}{\partial s} = h \frac{\partial \omega}{\partial \eta}$$

we get
$$\frac{\partial^2 \omega}{\partial \nu^2} = h^2 \frac{\partial^2 \omega}{\partial \xi^2} + h \frac{\partial h}{\partial \xi}\frac{\partial \omega}{\partial \xi}$$

and
$$\frac{\partial^2 \omega}{\partial s^2} = h^2 \frac{\partial^2 \omega}{\partial \eta^2} + h \frac{\partial h}{\partial \eta}\frac{\partial \omega}{\partial \eta}.$$

From these, it is apparent that G will be zero if simultaneously

$$\frac{\partial \omega}{\partial \xi} = \frac{\partial \omega}{\partial \eta} = \frac{\partial^2 \omega}{\partial \xi^2} = \frac{\partial^2 \omega}{\partial \eta^2} = 0.$$

Hence our problem is to find out a solution of the equation (4) which will satisfy the conditions that when $\xi = \alpha$

$$\omega = \frac{\partial \omega}{\partial \xi} = \frac{\partial \omega}{\partial \eta} = \frac{\partial^2 \omega}{\partial \xi^2} = \frac{\partial^2 \omega}{\partial \eta^2} = 0. \quad \ldots \quad (5)$$

4. Let
$$\frac{1}{\cosh 2\xi - \cos 2\eta}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\omega = \omega_1. \quad \ldots \quad (6)$$

Then from (4), we get

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\omega_1 = 8p_0(\cosh 2\xi - \cos 2\eta).$$

A solution of this equation is

$$\omega_1 = 2p_0(\cosh 2\xi + \cos 2\eta) - 2p_0\left(\cosh 2\alpha + \frac{\cosh 2\xi}{\cosh 2\alpha}\cos 2\eta\right).$$

The constant term has been so chosen as to make $\omega_1 = 0$ when $\xi = \alpha$.

From (6) then we get

$$\frac{1}{\cosh 2\xi - \cos 2\eta}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\omega =$$

$$2p_0\left[(\cosh 2\xi + \cos 2\eta) - \cosh 2\alpha - \frac{\cosh 2\xi}{\cosh 2\alpha}\cos 2\eta\right].$$

or $\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\omega =$

$$p_0\Big[(\cosh 4\xi - \cos 4\eta) - \frac{\cosh 4\alpha}{\cosh 2\alpha}(\cosh 2\xi - \cos 2\eta)$$

$$- \frac{1}{\cosh 2\alpha}(\cosh 4\xi \cos 2\eta - \cosh 2\xi \cos 4\eta)\Big].$$

The solution of this can be put in the form

$$\omega = \frac{p_0}{16}\left(\cosh 4\xi + \cos 4\eta\right) - \frac{p_0 \cosh 4\alpha}{4\cosh 2\alpha}(\cosh 2\xi + \cos 2\eta)$$

$$- \frac{p_0}{12\cosh 2\alpha}[\cosh 4\xi \cos 2\eta + \cosh 2\xi \cos 4\eta]$$

$$+ \frac{3p_0}{16}\cosh 4\alpha + \frac{p_0 \cosh 4\alpha}{3\cosh^2 2\alpha}\cosh 2\xi \cos 2\eta$$

$$+ \frac{p_0}{48}\frac{\cosh 4\xi}{\cosh 4\alpha}\cos 4\eta + \chi \qquad \ldots \quad (7)$$

where χ is a plane harmonic function to be so chosen as will make ω satisfy the boundary conditions (5).

We choose $\chi = p_0 \dfrac{\sinh^3 2\alpha}{2\cosh 2\alpha}(\xi - \alpha)$

$$- \frac{p_0}{3}\frac{\sinh^3 2\alpha}{\cosh^2 2\alpha}\sinh 2(\xi - \alpha)\cos 2\eta$$

$$+ \frac{p_0}{24}\frac{\sinh^3 2\alpha}{\cosh 2\alpha \cosh 4\alpha}\sinh 4(\xi - \alpha)\cos 4\eta.$$

The value of ω completed with the addition of this χ satisfies all the boundary conditions enumerated in (5).

ON THE DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2} + (A + 16\,q \cos 2\,nx)\,y = 0.$$

By Rama Shankar Varma, M.Sc.
Research Scholar, Allahabad University.

1. Introduction.

In a previous issue* of this journal, I have discussed the periodic solutions of the differential equation.

$$\frac{d^2y}{dx^2} + (A + 16q \cos 2nx)\,y = 0. \qquad \ldots \quad (1)$$

These periodic solutions, denoted by

$$c_ne_0\,(x, q),\ c_ne_1\,(x, q),\ c_ne_2\,(x, q) \ldots c_ne_m\,(x, q) \ldots\ldots$$

$$s_ne_1\,(x, q),\ s_ne_2\,(x, q) \ldots s_ne_m(x, q) \ldots\ldots$$

reduce to

$$1, \cos x, \cos 2x, \ldots \ldots \cos mx, \ldots$$

$$\sin x,\ \sin 2x, \ldots \ldots \sin mx, \ \ldots,$$

when the q in equation (1) reduces to zero, and hence the corresponding second solutions† may be taken so that they reduce to

$$x, \sin x, \sin 2x, \ldots \ \ldots \sin mx, \ldots$$

$$\cos x, \cos 2x, \ldots \ \ldots \cos mx, \ldots$$

when q becomes zero.

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* "On the Periodic solutions of the Differential Equation

$$\frac{d^2y}{dx^2} + (A + 16q \cos 2\,nx)\,y = 0,"$$

Vol. XVIII, No. 1, pp. 16—30. This paper will be referred here as Paper I.

† See the paper "On the Non-periodic solutions of the Differential Equation.

$$\frac{d^2y}{dx^2} + (A + 16q \cos 2\,nx)\,y = 0."$$

Allahabad University Studies, Vol. VI.

These second solutions have been denoted, in accordance with Whittaker's notation for the second solutions of Mathieu's equation, by

$$i_n\,\eta_0\,(x, q),\ \ i_n\,\eta_1\,(x, q),\ \ i_n\,\eta_2\,(x, q),\ \ldots\ \ldots\ i_n\,\eta_m\,(x, q), \ldots\ \ldots$$

$$j_n\,\eta_1\,(x, q),\ \ j_n\,\eta_2\,(x, q),\ \ldots\ \ldots\ j_n\,\eta_m\,(x, q)\ \ldots\ \ldots$$

The object of this paper* is to discuss the form of the equation and its solutions, and to point out an application of the differential equation to a problem in Tidal waves.

2. Form of the equation and its solutions.

The differential equation (1) occurs in many problems of Mathematical Physics. It contains three parameters, A, q and n, which can be reduced to two by changing the independent variable x to z by the relation $nx = z$; the equation (1) then assumes the form

$$\frac{d^2y}{dz^2} + \frac{1}{n^2}\,(A + 16q \cos 2z)\, y = 0.$$

Now a solution of (1) of order m corresponds to that value of A which tends to m^2 as q tends to zero. Hence this solution of (1), when expressed in terms of z, will be the same as a periodic solution of Mathieu Differential Equation (containing two parameters only), *viz.*

$$\frac{d^2y}{dz^2} + (A' + 16q' \cos 2z)\, y = 0, \qquad \ldots \quad (2)$$

only when m is a multiple of n. If m is not a multiple of n, the solution of (1) will give a function which is not a Mathieu Function†. The

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* This paper includes the results of my two papers, " On the solutions of the Differential Equation

$$\frac{d^2y}{dx^2} + (A + 16q \cos 2\,nx)\, y = 0 \text{''}$$

and. " On the application of the Differential Equation

$$\frac{d^2y}{dx^2} + (A + 16q \cos 2\,nx)\, y = 0,$$

and, in particular, (for $n = 1$) of Mathieu Differential Equation to a problem in Tidal Waves." the abstracts of which were read in the *Sixth Conference of the Indian Mathematical Society*, held at Nagpur.

† Whittaker and Watson : *Modern Analysis*, Chap. XIX.

solutions of equation (1) are, in this sense, more general than the Mathieu Functions, *viz.*, the even and odd periodic solutions of the equation (2).

It may be remarked that some writers* would call Mathieu Functions of fractional order such solutions of (1), as do not correspond to Mathieu Functions of integral order.

Again, from the integral equation

$$\psi(x) = B\cos mx + \frac{1}{m}\int_0^x (a' + 16q\cos 2nt)\sin m(t-x)\,\psi(t)\,dt, \quad \ldots \quad (3)$$

where $A = m^2 + a'$, and B is some constant, it was shown that

$$c_n e_m(x, q) = \cos mx + q\left\{\frac{2\cos(2n+m)x}{n(n+m)} + \frac{2\cos(2n-m)x}{n(n-m)}\right\}$$

$$+ q^2\left\{\frac{2\cos(4n+m)x}{n^2(n+m)(2n+m)} + \frac{2\cos(4n-m)x}{n^2(n-m)(2n-m)}\right\} + \ldots, \quad \ldots \quad (4)$$

subject to the condition that

$$A = m^2 - \frac{32}{n^2 - m^2}q^2 + \ldots, \quad \ldots \quad (5)$$

Here we notice that, if n is a sub-multiple of m, the solution breaks down. In such cases, the method used for the general case, applied direct, gives a valid solution, as we have seen in the papers referred to in the introduction to this paper.

For obtaining the solution of equation (1), for particular values of n and m in the case when n is a sub-multiple of m, say $m = kn$, we should, hence, substitute the given value of n in the expression for $c_n e_{k_n}(x, q)$. This avoids the difficulty which arises, on account of the vanishing of a factor in the denominator when we substitute the given value of n and m in $c_n e_m(x, q)$.

It may be noticed that, in paper (1) the solution $c_n e_o(x, q)$ was not deduced from the integral equation (3). We can, however, find it, if we

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* Messrs. Balth. Van der Pol and M. J. O. Strutt See their paper "On the stability of the solutions of Mathieu's equations" *Phil Mag.* (Series 6), Vol. 5, 1928.

take first the limit of $\dfrac{\sin m\,(t - x)}{m}$ as m tends to zero in (3), and then proceed as in the general case.

3. Application of the Differential Equation to a problem in Tidal Waves.

If η be the elevation of liquid, b and h the breath and the mean depth over the width b respectively, the equation of motion* for the tidal-wave propagation is :

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b} \frac{\partial}{\partial x}\left(bh \frac{\partial \eta}{\partial x} \right),$$

where b and h are functions of x.

In the case of simple harmonic motion, where η varies as $\cos(\sigma t + \varepsilon)$, the above equation becomes

$$\frac{g}{b} \frac{\partial}{\partial x}\left(bh \frac{\partial \eta}{\partial x} \right) + \sigma^2 \eta = 0 \qquad \ldots \quad (6)$$

Let us take †

$$\left. \begin{aligned} b &= b_0 \; (1 + q' \cos 2nx) \\ h &= h_0 / (1 + q' \cos 2nx) \end{aligned} \right\}, \qquad \ldots \quad (7)$$

so that the wave-length of the sinuosity of the breadth-curve of the canal is π / n. The quantity q' in (7) is assumed to be very small so that the disturbance by (i) the reflection of waves due to the transition of the section of the canal and (ii) the intereference in the tide due to the furrowed nature of the bottom may be greatly obviated.

Substituting the value of b and h as given by (7) in (6), and putting

$$\frac{\sigma^2}{g h_0} = k \,; \;\; kq' = 16q,$$

we have the differential equation

$$\frac{\partial^2 \eta}{\partial x^2} + (k + 16q \cos 2\, nx)\, \eta = 0, \qquad \ldots \quad (8)$$

which is similar to (1).

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* Lamb : *Hydrodynamics* (5th Edition), Art. 185.

† It may be remarked that a similar problem has recently been tackled by Messrs. A. C. Banerji and myself. *Vide* "On Tidal Waves in Canals having Sinuous Banks," *Pro. Camb. Phil. Soc*, Vol. XXIV, p. 567—77. In this paper, the breadth and depth curves were taken to be of the form $b = b_0\,(1 + q_1 \cos mx)$ and $h = h_0\,(1 + q^2 \cos nx)$ respectively.

We have seen that, for moderate values of q, the equation (8) admits of two kinds of periodic solutions, even and odd, with different definite values of k in each case, thus giving the periods of possible vibrations. The various oscillations in the canal will be evidently given by these solutions.

Thus the solution of order zero of the equation (8) gives the oscillation, without the time-factor,

$$\eta = B\, c_n e_0(x, q) = B\left[1 + \left(\frac{4q}{n^2} - \frac{28q^3}{n^6} + \dots\right)\cos 2nx \right.$$
$$+ \left(\frac{2q^2}{n^4} - \frac{160q^4}{n^8} + \dots\right)\cos 4nx$$
$$\left. + \dots \quad \dots \quad \dots \quad \dots \right],$$

with

$$k = -\frac{32}{n^2}q^2 + \frac{224}{n^6}q^4 + \dots,$$

where B is some constant.

Let us assume that at its mouth ($x = a$), the canal communicates with the open sea in which a tidal oscillation $\eta = C\cos(\sigma t + \varepsilon)$ is maintained. Hence restoring the time-factor, the complete solution is

$$\eta = C\,\frac{c_n e_0(x, q)}{c_n e_0(a, q)}\cos(\sigma t + \varepsilon),$$

where

$$\frac{\sigma^2}{gh_0} = k = -\frac{32}{n^2}q^2 + \frac{224}{n^6}q^4 + \dots$$

Similarly the even solution of order m of the equation (8) gives the tidal height, with the time-factor,

$$\eta = C\,\frac{c_n e_m(x, q)}{c_n e_m(a, q)}\cos(\sigma t + \varepsilon),$$

with

$$k = m^2 - \frac{32}{n^2 - m^2}q^2 + \dots \qquad \dots \quad (9)$$

where it is assumed as before, that the canal communicates, at its mouth $x = a$, with the open sea in which a tidal oscillation $\eta = C\cos(\sigma t + \varepsilon)$ is maintained.

Also, we have seen that q' must be small, hence the transcendental relation (9) between the speed σ and the quantity q' reduces to a simple polynomial, from which the value of σ can be easily determined. Again, since $\frac{\sigma^2}{gh_0} = m^2$ when $q' = 0$, we have only to take into consideration the value of k which converges to m^2.

The relation (9) also affords a very simple way of comparing the speed σ when the wave-length of the breadth curve sinuosity is varying.

Finally it may be remarked that we can take the breadth and depth curves in a more general form, *viz.*

$$b = b_0 \,(1 + \theta'_1 \cos 2x + \theta'_2 \cos 4x + \dots \dots)$$
$$h = h_0 / (1 + \theta'_1 \cos 2x + \theta'_2 \cos 4x + \dots \dots),$$

where $\theta'_1, \theta'_2 \dots \dots$ are all small.

Substituting these in equation (6), we have

$$\frac{\partial^2 \eta}{\partial x^2} + (\theta_0 + 2\theta_1 \cos 2x + 2\theta_2 \cos 4x + \dots)\, \eta = 0, \qquad \dots \quad (10)$$

where

$$\theta_0 = \frac{\sigma^2}{gh_0}, \quad 2\theta_n = \frac{\sigma^2}{gh_0}\, \theta'_n.$$

The differential equation (10) also occurs in connection the astronomical problem* of the mean motion of the Moon's perigee and has been discussed by Hill.

If F (z) be a periodic solution of (10) for a relation

$$f\,(\theta_0, \theta_1, \theta_2 \dots) = 0, \qquad \dots \quad (11)$$

between $\theta_0, \theta_1, \theta_2 \dots$, the complete solution, with the time-factor will in our case be

$$\eta = c\, \frac{F\,(z)}{F\,(a)} \cos(\sigma t + \varepsilon).$$

The value of the speed σ is given by the relation (11).

My thanks are due to Prof. A. C. Banerji, for the interest he has taken in the preparation of this paper.

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* *Acta Mathematica*, Vol. VIII, pp. 1 – 36 (1886). See also the papers on Hill's equation by Lindsay Ince, *Monthly Notices of the R. A. S.*, Vols. LXXV, LXXVI and LXXVIII.

ON THE REGULARIZATION OF THE PROBLEM OF THREE BODIES.*

By D. FERROLI, S.J.
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1. Introduction.

It is not my intention to enter here into a discussion of the general problem of three bodies, the first particular solutions of which were found by Lagrange before the year 1772, and were reproduced by Laplace in his Méchanique Céleste. To determine the mutual distances of the bodies Lagrange found three equations, of which one was of the third order, and the other two were each of order two, thus making the whole problem of the seventh order. No new solutions were discovered after those of Lagrange, until Hill developed his Lunar Theory and found solutions of greater practical value than those of Lagrange himself. The most important advance was made by Poincaré, who, assuming the masses of two of the bodies to be small compared to that of the third, proved that there were an infinite number of sets of initial conditions for which the motion was periodic. In 1899 Sir George Darwin assumed that one of the three masses was infinitesimal, and that the finite masses revolved in circles. By numerical examples several periodic orbits were discovered, whose stability was determined by methods analogous to those employed by Hill in his discussion of the motion of the lunar perigee. Darwin's curves, represented in bipolar co-ordinates by the equation

$$2\,\Omega = C$$

where C is a function of the masses and mutual distances of the bodies, are famous as representing barrier curves which cannot be crossed by trajectories characterized by the corresponding values of the constant of energy C. [cf. Moulton: *Celestial Mechanics*, pp. 319-20].

The question, however, was raised whether the problem could be regularized, *i.e.*, whether the systems of differential equations which

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* Read at the Madras Session of the Indian Science Congress, 1929.

embody it could be so transformed as to free them entirely from singularities in the entire field in which they exist. Some 35 years ago Painlévé initiated studies of this kind, and these were continued by Levi-Civita, Bisconcini, and especially by Sundmann. Here I shall indicate the methods of regularization which are due to Levi-Civita, basing my study chiefly on :

(1) Acta Mathematica ; 1916, Vol. 30, pp. 305—327;

(2) ,, ; 1918, Vol. 42, pp. 99—143;

(3) Rendiconti dei Lincei, 1915, Vol. XXIV, pp. 61—75;

(4) Questioni di Meccanica Classica e Relativista,
by T. Levi-Civita (Zanichelli, Bologna, 1924).

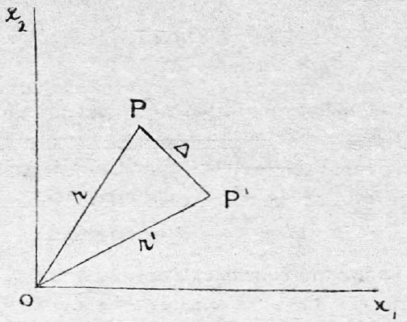
2. Sundmann's Results

Let the problem be limited to a plane, and let the bodies be O, P and P′, the mass of P being infinitesimal, and therefore having no influence on the motion of O or P′. We suppose O and P′ to rotate uniformly round their common centre of gravity (G). The problem can then be reduced to the study of the motion of P, which takes place in the plane on which O and P′ are rotating, under the influence of O and P′. Clearly there can only be a binary collision of P with O or with P′. Levi-Civita has succeeded in explicitly determining the condition of collision, which is unique and uniform, as had been affirmed by Painlévé.

Bisconcini analysed the binary collisions in the general problem of the three bodies, establishing two uniform relations characteristic of the collision, and confirming Painlévé's previsions. Sundmann then first examined the case of a general collision of the three bodies, and showed that this can take place only if the resultant moment of the quantities of motion of the three bodies is zero, (1907). In 1912 he succeeded in completely regularizing the problem of the three bodies, *i e.*, he succeeded in freeing the differential systems connected with the problem from all the singularities in the real field. The method which he followed, was however, indirect, and he obtained a regularized system which goes beyond the track the equations of General Dynamics. In 1915 Levi-Civita obtained a really dynamic regularization of the binary collisions in one plane, and it is his process that I shall develop here.

3. Limitation of the Problem to one Plane.

In the plane of the three bodies O, P, P′ consider a system of co-ordinates with the origin (in motion) at O and the directions fixed. Let



the co-ordinates of P and P′ be x_1, x_2, and x'_1, x'_2, and let the components of their momenta relative to the C. G. of O, P, P′ be

$$p_1,\ p_2 \text{ and } p'_1,\ p_2$$

respectively. Therefore the components of the analogous quantity of motion of O are

$$-(p_1 + p'_1),\ -(p'_2 + p'_2).$$

Let the masses of O, P and P′ be m_0, m, m', and let the distances OP, OP′, PP′ be r, r' and Δ.

If f is the constant of attraction, the force function is

$$U = f\left\{\frac{m_0 m}{r} + \frac{m_0 m'}{r'} + \frac{mm'}{\Delta}\right\}.$$

Let now

$$T = \frac{1}{2m}\left(p^2_1 + p^2_2\right) = \text{Kinetic Energy of P.}$$

$$T' = \frac{1}{2m'}\left(p'^2_1 + p'^2_2\right) = \text{K. E. of P}'.$$

$$T_0 = \frac{1}{2m_0}\left((p_1 + p'_1)^2 + (p'_2 + p_2)^2\right) = \text{K. E. of O}$$

$$\tau = T + T' + T_0;\ H = \tau - U.$$

clearly H has only the singularities of U (which correspond to zero values of r, r' and Δ) and the singularities which arise from p_1, p_2 and p'_1, p'_2 assuming infinite values. As is well-known, the canonical equations of motions are

$$(1) \qquad \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \text{ etc.} \qquad (i = 1, 2).$$

Since the second members are derived functions of H, they have the singularities of H at most. We notice also that the singularities arising from p_1, p_2 ... assuming infinite values, are dependent on the singularities which arise from r, r', Δ assuming zero values. For since

$$H = T - U = \text{constant},$$

we have that, as long as U is finite (*i.e.* r, r' Δ, $\neq 0$), T, and therefore also the p's remain finite. Again, as long as M, *i.e.*, the resultant moment of the quantities of motion does not vanish, a general collision is impossible. Also, for a binary collision, say of P and O, we have for $t \to t_1$ (the instant of the collision)

$$(a) \qquad \underset{t \to t_1}{\mathrm{Lt}}\ r = 0$$

(*b*) x' and p' tend to finite limits for $t \to t_1$. Therefore

$$\underset{t \to t_1}{\mathrm{Lt}}\ r' = \underset{t \to t_1}{\mathrm{Lt}}\ \Delta > 0.$$

(*c*) $\frac{1}{r} \to \infty$ for $t = t_1$, but remains integrable, so that, putting $du = \frac{dt}{r}$, we define a parameter u which grows with t and tends to a finite value u_1 when $t \to t_1$.

$$(d) \quad \underset{t \to t_1}{\mathrm{Lt}}\ r\,T = \underset{t \to t_1}{\mathrm{Lt}}\ r\,(U + H) = \underset{t \to t_1}{\mathrm{Lt}}\ f\,m_0 m$$

(for r H = 0 when $r = 0$).

Similarly p'_1, p'_2 etc. multiplied by r tend to zero.

$$\therefore \underset{t \to t_1}{\mathrm{Lt}}\ r\,T = \underset{t \to t_1}{\mathrm{Lt}}\ r \left\{ \frac{1}{2m} + \frac{1}{2m_0} \right\} \left(p_1^2 + p_2^2 \right) = f\,m_0 m.$$

4. Regularization of the Problem in one Plane.

If to H we add a constant, the equations (I) remain unaltered. Remembering that they admit the integral H = a const., let us substitute (H — E) for H and fix our attention on those solutions which correspond to a determinate value of E. Let now

$$(2) \qquad H^x = r\,(H - E) = r\,T + r\,(T' + T_0) - r\,U - r\,E.$$

For t substitute u, where $r.du = dt$.

Then equations (I) become

$$(1') \qquad \frac{dx_i}{du} = \frac{\partial H^x}{\partial p_i}; \quad \frac{dp_i}{du} = -\frac{\partial H^x}{\partial x_i} \qquad (i = 1, 2), \text{ etc.}$$

We are however, interested only in those solutions for which $H^x = 0$, and which correspond to H = E.

If we observe the structure of H^x near a binary collision P, O, we notice an improvement on the original characteristic function. In fact the infinites have disappeared. Yet the function is not regular, for there is a critical value for x_1 and x_2 equal to zero. Again, owing to (d), $p_1^2 + p_2^2 \to \infty$. Hence the collision is not within the field of regularity. To obtain the complete regularization, and at the same time keep the canonical form of the differential system, we shall express x_1 and x_2, in terms of ξ_1 and ξ_2 where:

$$(3) \qquad x_1 + \sqrt{-1}\, x_2 = (\xi_1 + \sqrt{-1}\, \xi_2)^2;$$

also let

$$(4) \qquad p_1 dx_1 + p_2 dx_2 = \pi_1 d\xi_1 + \pi_2 d\xi_2$$

The conditions (4) will ensure the preservation of the canonical form.

More explicitly, we may assume

$$(5) \qquad p_1 + \sqrt{-1}\, p_2 = \frac{\pi_1 + \sqrt{-1}\ \pi_2}{2\,(\xi_1 - \sqrt{-1}\ \xi_2)}$$

from which (4) may easily be deduced.

Again, putting $\rho = |\xi_1 + \sqrt{-1}\ \xi_2|$,

we have, since $|x_1 + \sqrt{-1}\, x_2| = r,$

$$(6) \qquad r = \rho^2 = \xi_1^2 + \xi_2^2 \qquad \text{from (3).}$$

Also, equating the squares of the moduli, in (5) we get

$$(7) \qquad p_1^2 + p_2^2 = \frac{\pi_1^2 + \pi_2^2}{4\rho^2}.$$

Multiplying both sides of (5) by

$$r = \{\xi_1 + \sqrt{-1}\ \xi_2\}\{\xi_1 - \sqrt{-1}\ \xi_2\}$$

we get

$$(8) \quad r(p_1 + \sqrt{-1}\ p_2) = \tfrac{1}{2}(\pi_1 + \sqrt{-1}\ \pi_2)(\xi_1 + \sqrt{-1}\ \xi_2)$$

whence

$$\frac{r^2}{2}(p_1 + \sqrt{-1}\ p_2)^2 = \frac{(\pi_1 + \sqrt{-1}\ \pi_2)^2}{8}(\xi_1 + \sqrt{-1}\ \xi_2)^2$$

i.e., $$\frac{r}{2}(p_1^2 + p_2^2) = \frac{\pi_1^2 + \pi_2^2}{8} = \frac{f(m_0^2 m^2)}{m_0 + m_1} \qquad [\text{(from } d)].$$

Hence $\pi_1^2 + \pi_2^2$ remains finite when P and O impinge against each other.

Also, since the direction of OP tends, in general, to a definite limit, the same happens for π_1 and π_2 separately. From the above equations we see that x_1, x_2, r, $r\ (p_1^2 + p_2^2)$, rp_1 and rp_2 are all functions of the second degree of the new variables ξ_1 and ξ_2, π_1 and π_2, and therefore their regularity in the neighbourhood of $\xi_1 = \xi_2 = 0$, and of the finite values of π_1 and π_2 which belong to a collision, is ensured.

Remembering now that $\frac{1}{r}$, $\frac{1}{\Delta}$ are regular in the neighbourhood of a collision with respect to x_1 and x_2, clearly this characteristic is preserved also when ξ_1 and ξ_2 are substituted for x_1 and x_2.

Since in $$\mathrm{H}^x = r\,(\mathrm{T} - \mathrm{U} - \mathrm{E}),$$

x_1, x_2, p_1 and p_2 appear only through r, $r\ (p_1^2 + p_2^2)$ and rp_1, rp_2, we see that the complete regularization of H^x has been attained, and conse-

quently also of the corresponding differential equations, which are

$$\left.\begin{aligned} \frac{d\xi_i}{du} = \frac{\partial H^x}{\partial \pi_i}; \quad \frac{u\pi_i}{du} = -\frac{\partial H^x}{\partial \xi_i} \\ \frac{dx'_i}{du} = \frac{\partial H^x}{\partial p_i}; \quad \frac{dp_i}{du} = -\frac{\partial H^x}{\partial x_i} \end{aligned}\right\} i = 1, 2.$$

5. **The Problem in Space.**

In space we have

$$T = \frac{1}{2m} [p_1^2 + p_2^2 + p_3^2]$$

$$T' = \frac{1}{2m'} [p'_1{}^2 + p'_2{}^2 + p'_3{}^2]$$

$$T_0 = \frac{1}{2m_0} \{ (p_1 + p'_1)^2 + (p_2 + p'_2)^2 + (p_3 + p'_3)^2 \}$$

Levi-Civita tried in vain to obtain the solution by means of point-transformations, where ξ_1, ξ_2, ξ_3, and π_1, π_2, π_3 are substituted for $x_1, x_2, x_3, p_1, p_2, p_3$, and the relation:

(9) $p_1dx_1 + p_2dx_2 + p_3dx_3 = \pi_1 d\xi_1 + \pi_2 d\xi_2 + \pi_3 d\xi_3 + dW$

is valid, W being an arbitrary function. Then he tried contact transformations, and in particular those furnished by Jacobi's method of integration. To understand the geometrical and kinematical significance of the new variables (ξ_i, π_i) in place of (x_i, p_i), $(i = 1, 2, 3)$, let us consider, at a particular instant t, the point P and the vector q which are respectively defined by the values of x_i and p_i corresponding to t. In reality q represents the momentum of P relatively to the C. G. (G) of the three bodies. But, along with the real motion, we can also consider a hypothetical movement, which may be called *intermediate*, in which q represents the momentum of P with respect to O (not G), and in which the force acting on P is reduced to the attraction of the centre O whose potential is fm_0m/r.

If, as is always the case for the planets,

$$T - \frac{fm_0m}{r} = \frac{1}{2m} q^2 - \frac{fm_0m}{r} < 0$$

then, the hypothetic movement round O, defined by the movement P, q, is Keplerian, and the classical transformations of Jacobi and Poincare are those in which corresponding elements of this Keplerian movement are substituted for x_i, p_i.

The criterion of referring the real phenomenon to hypothetical movements which have in common the motion P, q, can be variously modified. Thus, whenever E (the total energy of the real motion) is negative, we can imagine a new constant k defined by the values x_i, p_i at the instant in question, through the equation

$$\frac{1}{2m} q^2 - \frac{k}{r} = \mathrm{E}$$

and consider the Keplerian movement which would take place starting rom P, q, under the potential k/r which is substituted for $f_0\, m_0 m/r$. The elliptic elements corresponding to this hypothetical movement are called *isoenergetic* by Levi-Civita, whilst the ordinary elements he calls *isodynamic*. However, not even the isoenergetic elements do regularize the problem.

6. Canonical transformation taken from parabolic motion. Regularization of a Binary Collision.

If
$$\frac{1}{2m} q^2 - \frac{k}{r} = 0,$$

the hypothetical movement corresponds to an intermediate parabolic orbit. Its focus is O. Now we can define two vectors, one a length vector

$$\xi = 2k$$

from the focus to the vertex and another π parallel to q and whose length is given by

$$\pi = \frac{r}{\xi} q.$$

If ξ_i, π_i are the components of ξ and π respectively, and if

$$-\tfrac{1}{2}\mathrm{W} = \sum_1^3{}_i\, \xi_i\, \pi_i,$$

it can be shown that

$$(10) \quad x_i = \pi^2 \xi_i + W\pi_i;\; p_i = \frac{\pi_i}{\pi^2}; \qquad (i = 1, 2, 3).$$

Since from the last equation the previous one can be derived, we see that the equations (10) define a contact transformation. When $P \to 0$, $q \to \infty$ and rq^2 remains finite and different from zero. Therefore $2k$ which is equal to rq^2/m, remains finite also when P collides with O. Hence ξ_1, ξ_2, ξ_3 tend to finite limits. The π_i on the contrary tend to zero, for rq/ξ, which is equal to π tends to zero, and the π_i are the components of π. We thus notice that, not only the x_i, as is clear from (10), are regular functions of ξ_i and π_i in the neighbourhood of a collision (*i.e.* for the π_i's equal to zero, and for values of the ξ_i's which do not all vanish) but that the same is true of

$$(11) \quad r = \xi \pi^2;\; rp_i = \xi \pi_i;\; r(p_1^2 + p_2^2 + p_3^2) = \xi.$$

Hence analogously with the problem in one plane, we see that H^x and therefore the differential system is regularized.

7. Symmetrical Parameter. Complete Regularization.

What has been said of the collision PO, applies with the necessary changes, to the collisions P′P and P′O. For the collision PO we have employed a contact transformation, and substituted u for t according to the formula

$$du = \frac{dt}{r}.$$

Now, remembering that the product

$$rU = fm_0 m + f\frac{m_0 m'}{r'} r + f\frac{mm'}{\Delta} r$$

considered as a function of ξ_i, π_i is regular even when the collision PO takes place, and does not vanish, it is clear that we can substitute in place of u a new variable τ defined by the relation

$$(12) \qquad d\tau = r\,U du, \qquad \text{or by} \qquad d\tau = U dt. \qquad (12')$$

τ has over u the advantage of being symmetrical with respect to the three bodies. If in (1) we substitute τ for u and put

$$\text{(13)} \qquad F = \frac{1}{rU} H^x, \text{ or } F = \frac{1}{U}(H - E)$$

noticing that, for those solutions in which we are interested F vanishes with H^x, the system (1) in space can be written as follows

$$\text{(14)} \qquad \left\{ \begin{aligned} \frac{dx_i}{d\tau} &= \frac{\partial F}{\partial p_i}, & \frac{dx'_i}{d\tau} &= \frac{\partial F}{\partial p'_i} \\ \frac{dp_i}{d\tau} &= -\frac{\partial F}{\partial x_i} & \frac{dp_i}{d\tau} &= -\frac{\partial F}{\partial x'_i} \end{aligned} \right\} \qquad (i = 1, 2, 3)$$

By adopting the parameter τ we are sure that the local regularization of the binary collisions does not require any modification of independent variable, but only contact transformations of the type already indicated, and eventually cartesian transformations, to shift the origin from O to P or to P'.

Remebering that, according to Sundmann's Theorem, no general collision can take place unless M, the resultant moment of momentum, vanishes, we can say that the canonical system (14) is regular, or can easily be made regular by a simple canonical transformation of the unknown functions for any value of τ, *i.e.* of the time, even in case a collision should eventually take place.

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ON CLOSED FORMS AND POLAR FORMS.

By R. Vaidyanathaswamy.

1. In the paper 'On the Rank of the Double-binary Form' (*Proceedings of the London Mathematical Society*, Series 2, Vol. 94, Part 2, p. 83), the rank of an (m, n) form $a_x{}^m b_y{}^n$ was defined as the rank of the matrix of $m+1$ rows and $n+1$ columns, formed by its coefficients. Further, if

$$F(x, y) = a_x{}^m b_y{}^n = a_{0x}{}^m b_{0y}{}^n = a_{1x}{}^m b_{1y}{}^n = \quad \dots \quad ,$$

then the rth rank covariant $G_r(x, y)$ of $F(x, y)$ is given by

$$G_r = (a_0 a_1 \dots a_r)(b_0 b_1 \dots b_r)\, a_{0x}^{m-r}\, a_{1x}^{m-r} \dots a_{rx}^{m-r}\, b_{0y}^{n-r} \dots b_{ry}^{n-r},$$

where

$$(a_0 a_1 \dots a_r) = (a_0 a_1) \dots (a_0 a_r)(a_1 a_2) \dots (a_1 a_r) \dots (a_{r-1} a_r),$$

$$(b_0 b_1 \dots b_r) = (b_0 b_1) \dots (b_0 b_r)(b_1 b_2) \dots (b_1 b_r) \dots (b_{r-1} b_r);$$

and the necessary and sufficient condition that $F(x, y)$ may be of rank r is $G_{r-1} \neq 0$; $G_r \equiv 0$. In particular the first rank covariant

$$G_1 = (a_0 a_1)(b_0 b_1)\, a_{0x}^{m-1}\, a_{1x}^{m-1}\, b_{0y}^{n-1}\, b_{1y}^{n-1}$$

is the (1,1) transvectant of F, and its vanishing is the condition that F may be the product of two binary forms in x and y respectively.

Further, for the special case in which $m = n$, and the form F has its maximum rank $n+1$ (so that the invariant $G_n \neq 0$), the series of rank covariants

$$G_0 (= F),\ G_1,\ \dots,\ G_{n-1}$$

display a certain reciprocity; namely, if the penultimate rank covariant G_{n-1}, which is a form of order (n, n) in (x, y), be taken as our fundamental form, and *its* successive rank covariants be formed, we only obtain the same series in reverse order, multiplied by scalar factors; in particular, $G_0 = F$ is, apart from a scalar factor, the penultimate rank covariant of G_{n-1}.

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NOTES AND QUESTIONS.

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Notes and Questions.

On the History of our common system of numerical notation.

The present article aims to make a contribution to the history of mathematical ideas by seeking a reasonable explanation of the late adoption of our present system of numerical notation, with its local values and operational 0, notwithstanding its great simplicity and the high states of civilization reached by various nations before its adoption. Among these nations the ancient Greeks are especially conspicuous in view of their remarkable mathematical attainments and their adoption of an alphabetical numerical notation after they had used another system known as the Attic system. One might have thought that since they used two systems of numerical notation they would have been led to consider the advantages of such a system as we now possess, which requires a much smaller number of distinct symbols than were needed in their alphabetical system even for numbers which are less than one hundred.

While our common system of numerical notation employs only ten distinct symbols it should be noted that all of these symbols are needed to represent the first ten natural numbers. It is well-known that in many of the older systems the first nine natural numbers were represented by from one to nine strokes respectively and hence these small numbers, which naturally presented themselves most frequently in the early stages of civilisation, required only one type of symbols. This method of representing numbers which are less than ten was employed by both of the nations which seem to have made the greatest mathematical advances before 1000 B.C., *viz.*, the ancient Egyptians and the ancient Sumerians. While the Greeks did not follow these ancient practices in their alphabetical system of numerical notation they did not require in this system any new symbols for these nine numbers since they employed therefor the well-known letters of their alphabet.

Hence we have arrived at the first serious difficulty involved in our common system of numerical notation as regards its early adoption, *viz.*, that ten distinct symbols are needed even when only the first ten natural

numbers are used and that these symbols had to be learnt only because they were needed in the numerical notation. None of the earlier systems of numerical notation made such great demands right at the start. There is, however, another element in our common system of numerical notation which was a much more serious obstacle to its early adoption, *viz.*, the operational 0 involved therein. It is obvious that a local value system of numerical notation can be so defined that only the emptiness 0 is needed for its development, but such a system does not present many of the greatest advantages inherent in our modern system. An emptiness 0 was found in various other systems of numerical notations. The Babylonians employed such a 0 several centuries before our era, and the Greeks made use of such a 0 at least as early as the time of Ptolemy in writing their sexagesimal fractions. Such a 0 was also used by the Maya of Central America.

The development of numerical notations cannot be understood unless we bear in mind that they serve both for recording results and for operating with numbers. As far as we know now the former of these purposes was the only one considered in the early development of these notations and it seems likely that while the operational advantages of our modern system are often especially emphasized they were not considered when this system first began to be used. An abacus may have served similar purposes to a local value numerical notation with an emptiness 0, and the use of the abacus evidently enables one to get along with a recording system of numerical notation. In order to understand the development of numerical notations we must therefore bear in mind that *numerical notations were invented with a view to recording numbers and not with a view to operating therewith.*

The operational properties of the nine digits 1, 2, ... , 9 were naturally noted long before distinct and independent symbols for these digits were employed, but the operational properties of 0 are very different therefrom and consequently began to be commonly used at a much later date. As regards addition these properties are similar to those of the identity in our modern group theory, but it should be noted that the explicit formulation of the group concepts came late in the development of our subject notwithstanding their great simplicity and their wide generality. The properties of 0 as regards addition are, however, its simplest operational properties and we find that Nicomachus already

noted that 0 added to 0 gives 0, and that these properties were later formulated much more clearly in India. The operational properties of 0 as regards multiplication are much more striking in view of the fact that every finite number is transformed into 0 by this operation with 0.

The difficulties involved in using 0 with its unique operational properties can perhaps be more fully appreciated if we think of its varied fortunes as regards division in recent times. Some of the older mathematicians made free use of 0 as a divisor by means of the postulate that every other number is thus transformed into infinity, while the result of dividing it by itself is indeterminate just as the product of 0 and infinity is indeterminate. On the other hand, many of the modern writers prefer to exclude the operation of dividing by 0 entirely and to say that 0 has no operational properties as regards division. It is obvious that the question of admitting or excluding the operation of dividing by 0 is a question of postulates, and that from a historical point of view it may be said that progress has usually been effected by admitting that the newer numbers have also per ational properties.

In view of these difficulties relating to an operational 0, it is more easy to understand why the ancient Greeks did not develop a positional arithmetic in the modern sense of this term and why they would naturally not have been particularly impressed by the steps in this direction which had been taken early by the ancient Babylonians, in case they became familiar with these steps during their early travels. This becomes the more plausible if we recall that some of the early Greek writers regarded it as desirable to exclude also rational fractions from the numbers which should enjoy full rights as numbers. There is obviously a wide difference between the emptiness 0 and an operational 0, and in many cases it is difficult to determine whether early writers who employed a symbol for 0 had in mind merely the emptiness 0 or the operational 0 of our times. If we assume that the term 'positional arithmetic' implies that the 0 involved therein is the latter 0 then it follows that this arithmetic is of comparatively recent origin, since 0 was not finally regarded as a number before the seventeenth century of our era. [J. Tropfke, *Geschichte der Elementar-Mathematik*, Volume 2, 1921, page 65.]

It should be explicitly noted that even if the term 'positional arithmetic' is used with the meaning of an emptiness 0 it cannot be said

that the ancient Greeks employed such an arithmetic since their emptiness 0 was used only in special cases. On the other hand, this emptiness 0 may have exerted a great influence in the evolution of our modern positional arithmetic in view of the fact that the works of Greek scholars in general, and of Ptolemy in particular, were widely read at about the time when our modern positional arithmetic began to be used. It should also be noted that it is difficult for us, who are so familiar with the common positional arithmetic to compare its operational merits with those of the alphabetical numerical notation employed by the Greeks. The noted French mathematical historian, Paul Tannery, stated in his *Memories Scientifiques*, Volume 1, page 101, that he drilled himself in calculating with Greek letters and that he was surprised by the advantages which this system presented after becoming familiar with it in a practical way. In particular, he went through all the calculations contained in the *Measurement of the Circle* by Archimedes and found no appreciable difference between the times required to perform them by means of the Greek numerical letters and by our common symbols.

While these experiments by Tannery may indicate that we are inclined to over-estimate the advantages of our modern system of numerical notation, these advantages obviously become more and more apparent as we use larger and larger numbers, or employ a larger and larger number of decimal places to secure the desired degree of accuracy. We could not tolerate a system of numerical notation in our day which calls for an endless number of new symbols to represent the higher powers of the base. The facts about our positional arithmetic therefore seem to be about as follows: It appears now so simple and useful that it is, at first, a surprise to us to find that its general adoption was delayed so long notwithstanding the early steps in its direction. This delay seems to be most satisfactorily explained by the difficulties involved in an operational 0 and the need of so many distinct and independent symbols even when dealing with the natural numbers which do not exceed ten. There seems to have been a tendency to use relatively large numbers, such as one hundred, one thousand, ten thousand, etc., long before all the intermediate numbers were fully understood. Hence special symbols began to be used for such large numbers and it was not an easy matter to discard these special symbols after they had secured a firm hold on the people.

There was also a certain flexibility as regards the use of different bases which had to be abandoned when a system involving only powers of a single base was adopted For instance, the ancient Hindus and the ancient Chinese are said to have had a secondary base which is equal to 10^{53}; [J. Tropfke, *Geschichte der Elementar-Mathematik*, Volume 1, 1921, page 5.] The well-known "Sand-reckoner" of Archimedes also exhibits a much more compact form of denoting very large numbers than would be possible by our ordinary system of numerical notation. These considerations may suffice to exhibit the reasonableness of the late appearance of our now common system of numerical notation with its operational 0. This is merely one side of the history of this system. The recent article by B. Datta on the "Jaina School of Mathematics," published in the *Bulletin of the Calcutta Mathematical Society*, Volume 21, relates to quite a different side of this history.

In his address before the Paris International Congress of Mathematicians, 1900, M. Cantor directed attention to the desirability of a history of the mathematical ideas which run through the various branches of our subject. Such a history would naturally aim to seek an explanation of the historical facts since one of the greatest incentives for mathematical studies has always been the securing of harmony in our thinking. In particular, if our modern system of numerical notation is superior to all those which preceded it, one would naturally desire to know what elements involved therein prevented its earlier adoption. One is thus led to see some of the weak points as well as the strong points in the earlier mathematical developments, and a history of our subject is not ideal unless it aims to be equally interested in both of these sides of the earlier developments of our subject. The mathematical attainments of the earlier civilizations constitute an important element of their relative grades, and when one finds a mixture of errors and correct results in their literature both types of results should receive attention. Otherwise the picture is misleading.

University of Illinois. G. A. MILLER.

An elementary note on Waring's Theorem on Cubes.

1. By making use of the fact that the double ofevery odd positive integer is a sum of *three* squares, Baer* proved in 1913 that every large number 4096 u, where u is odd, is a sum of the cubes of seven positive integers. It seems interesting to know how much can be found in this direction, if we use the elementary result that every positive integer is a sum of four squares. Using this fact only, I prove in this note that every large number of the form 4096 u, where u is odd, is a sum of nine cubes, all of which can be made to exceed an arbitrary number by taking u sufficiently large. The proof follows that of Baer.

2. Let u be odd and greater than $(8 . 2^{15} + 2^6)\, 2^6$; then we can always find a positive integer v ($\geqslant 2$) such that

$$(8 . 2^{15} + 2^6) . 2^{3v} < u < (8 . 2^{15} + 2^6) . 2^{3(v+1)}$$

Put

$$u - 8 . 2^{15} . 2^{3v} = t,$$

so that t is odd, and

$$2^{3v+6} < t < 2^{3v} (8 . 2^{15} . 7 + 8 . 2^6)$$

$$< 57 . 2^{3v+15}.$$

Let

$$t = \rho^3;$$

then

$$\rho > 2^{v+2}.$$

Also,

$$0 = t - \rho^3 \leqslant t - [\rho]^3 < t - \{ [\rho] - (3 . 2^v - 1) \}^3$$

$$< t - \{ \rho - 3 . 2^v \}^3 = 9 . 2^v \sqrt[3]{t^2} - 27 . (2^{2v}) . (\rho - 2^v)$$

$$< 147 . 2^{3v+10},$$

where $[x]$ is the greatest integer contained in x.

Now let n stand for any of the $3 . 2^v$ positive integers

$$[\rho],\ [\rho] - 1, \ldots, [\rho] - (3 . 2^v - 1);$$

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* *Math. Ann.* Vol.74, pp. 511–514.

then, from the above inequalities,

$$0 \leqslant t - n^3 < 147 \cdot 2_{3v+10} \qquad \dots \quad (1)$$

Now let

$$\theta = t - 3 \cdot 2^{v-1}.$$

where $v \geqslant 2$, and θ is an odd integer. Now every odd number is congruent (mod. $3 \cdot 2^v$) to the cube of some number, necessarily odd. For, let α and β be two odd positive integers less than $3 \cdot 2^v$, and, if possible, let

$$\alpha^3 - \beta^3 \equiv 0 \pmod{3 \cdot 2^v},$$

i.e.,

$$(\alpha - \beta)(\alpha^2 + \beta^2 + \alpha\beta) \equiv 0 \pmod{3.2^v}.$$

Now, $\alpha^2 + \beta^2 + \alpha\beta$ is an odd number; so, there are only two possibilities; either

$$\alpha - \beta = 2^v, \quad \alpha^2 + \beta^2 + \alpha\beta \equiv 0 \pmod 3;$$

or

$$\alpha - \beta = 2.2^v, \quad \alpha^2 + \beta^2 + \alpha\beta \equiv 0 \pmod 3)$$

since

$$\alpha^2 + \beta^2 + \alpha\beta = \tfrac{1}{4} \{ (2\alpha + \beta)^2 + 3\beta^2 \},$$

it is easy to see that neither case is possible.

It follows that we can choose an odd number ϕ from the set

$$[\rho], [\rho] - 1, \dots, [\rho - (3.2^v - 1)],$$

such that

$$\theta - \phi^3 \equiv 0 \pmod{3.2^v}$$

Hence

$$t = \phi^3 + 3.2^{v-1}(2s + 1)$$

$$= \phi^3 + 3\, 2^{v-2} . \mathrm{M},$$

where s and M are integers. From (1) we see that

$$0 \leqslant \mathrm{M} < 49 \cdot 2^{2v+12}.$$

Hence, since M is a non-negative integer, we have

$$\mathrm{M} = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

where

$$0 \leqslant x_s < 7 \cdot 2^{v+6}, \ (s = 1, 2, 3, 4).$$

Hence

$$4096u = 2^{12}.(t + 8.2^{15}.2^{3v})$$
$$= 2^{12}(\phi^3 + 3.2^{v-2}M + 8.2^{15}.2^{3v})$$
$$= (2^4.\phi)^3 + \sum_{s=1}^{s=4}\{(2^{v+9} + x_s)^3 + (2^{v+9} - x_s)^3\}$$
$$= \sum_{k=1}^{9}(\phi_k)^3,$$

where the ϕ's are positive integers which tend to infinity with u.

Our theorem is now proved.

3. *Literature.*

(1) A. Wieferich. "Beweis des Satzes, dass sich eine jede ganze Zahl als Summe von höchstens neun positiven Kuben darstellen lässt" *Math. Annalen*, **66** (1909), pp. 95—101.

(2) E. Landau. "Über eine Anwendung der Primzahltheorie auf das Waringsche Problem in der elementaren Zahlentheorie," *ibid*, pp. 102—105.

(3) E. Landau. *Archiv Math. Phys.* (3), **18**, 1911, pp. 248—252.

(4) W. S. Baer. "*Beiträge zum Waringschen Problem*," Dissertation, Göttingen, 1913, 74 pp.

(5) W. S. Baer. "Über die Zerlegung der ganzen Zahlen in sieben Kuben," *Math. Annalen*, **74** (1913), pp. 511—514.

(6) L. E. Dickson. "A generalization of Waring's Theorem on Cubes," *Bulletin Amer. Math Soc.*, **33**, 1927, pp. 299-300.

S. D. CHOWLA.

A Property of Positive odd Integers.

Dubois* has proved that every odd number > 41 is a sum of four positive squares. The object of this note is to prove the better result :

Every large odd number n is equal to $x_1^2 + x_2^2 + x_3^2 + x_4^2$, where the x's are all greater than $n^{\frac{1}{2}-\varepsilon}$, and ε is arbitrarily small and positive .

Let $f_s(n)$ denote the number of distinct ways of expressing n in the form

$$x_1^2 + x_2^2 + \ldots + x_s^2,$$

where the x's are all $\geqslant 0$. Thus $f_2(25) = 2$, $f_4(7) = 1$, and so on Now if n is odd, the co-efficient of x^n in

$$(\sum_{-\infty}^{\infty} x^{n^2})^4$$

is 8 times the sum of the odd divisors of n. From this it easily follows that

$$f_4(n) > \frac{n}{48} \qquad \ldots \quad (1)$$

Again, we know that

$$f_2(n) \leqslant \tfrac{1}{2}\{1 + d(n)\}, \qquad \ldots \quad (2)$$

where $d(n)$ is the number of divisors of n.

Hence

$$f_3(n) \leqslant \tfrac{1}{2}\{1 + D(n)\}\ \{1 + \sqrt{n}\}, \qquad \ldots \quad (3)$$

where $D(n)$ is the largest of the numbers $d(1)$, $d(2)$, ... $d(n)$.

Again, if θ is the greatest integer contained in $n^{\frac{1}{2}-\varepsilon}$, then the number of distinct ways in which

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad (x\text{'s all} > \theta)$$

is certainly greater than

$$f_4(n) - \sum_{s=0}^{\theta} f_3(n - s^2).$$

From (1) and (3) it follows that this expression is greater than

$$\frac{n}{48} - (n^{\frac{1}{2}-\varepsilon} + 1)\,\tfrac{1}{2}\,(1 + \sqrt{n})\,(1 + D(n)).$$

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* *L'intermediare des maths*, 18, 1911, pp. 55-6, 224-5.

Since $D(n)$ is of lower order than any power of n, this expression tends to infinity with n. In particular, there is at least one representation of n by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2$$

with all the x's $> \theta$. This is the theorem stated.

S. D. CHOWLA.

The Extension of Salmon's Theorem to the Conic.

1. Since Salmon's own proof of the theorem associated with his name consists in relating the symmetrical expression $x_A x_B + y_A y_B - a^2$, on the one hand to the distance between the point A and the line

$$x_B x + y_B y - a^2 = 0,$$

and on the other hand to the distance between the point B and the line

$$x_A x + y_A y - a^2 = 0,$$

it is only fair to suggest that the natural extension to the conic $ax^2 + by^2 = 1$ is a theorem depending in the same way on the symmetry of the expression $ax_A x_B + by_A y_B - 1$. Whether the axes are rectangular or oblique, the quotient $(lx + my + n)/n$ is the ratio of the distance to the line $lx + my + n = 0$ from the point (x, y) to the distance to the same line from the origin. If then, as in the figure on p. *37* of the *Notes and Questions* section of the current volume of the *Journal* (see below) AP, CY are the perpendiculars on the polar of B, and BQ, CX are the perpendiculars on the polar of A, we can equate $-(ax_A x_B + by_A y_B - 1)$ to both AP/CY and BQ/CX, and therefore these two ratios are equal. That is to say, the immediate extension of Salmon's theorem is to be recognised less in Sawayama's result than in the second of the two results which Krishnaswami Aiyangar combines to obtain Sawayama's.

Let us apply the same analytical arguments to the general conic $S = 0$. If OL, OM are the perpendiculars from the origin of co-ordinates on the polars of A and B, we have, with the usual notation,

$$AP/OM = S_{AB}/(gx_B + fy_B + c),\ BQ/OL = S_{AB}/(gx_A + fy_A + c).$$

But the line $gx + fy + c = 0$ is the polar of the origin, and the quotient

$$(gx_A + fy_A + c)/(gx_B + fy_B + c)$$

is the ratio of the distances to this line from A and from B. That is to say, if AF, BG are the perpendiculars from A, B on the polar of O, the theorem suggested by the symmetry of S_{AB} is

$$\frac{AP \cdot OL}{OM \cdot BQ} = \frac{AF}{BG}.$$

To see the structure of this theorem, we have only to change the notation. In relation to the conic, the points A, B, O are equally arbitrary, and the distances involved are the distances of each point from the polars of the other two. Replacing O by C, which is not now supposed to be the centre of the conic, we can express the result in the simple form of

THEOREM 1. *If P_q denotes the distance from a point P to the polar of a point Q with respect to a given conic, then if A, B, C are any three points,*

$$\frac{B_a}{C_a} \cdot \frac{C_b}{A_b} \cdot \frac{A_c}{B_c} = 1.$$

In this result there is no actual ambiguity, since distances from the same line are naturally to be measured in the same direction.

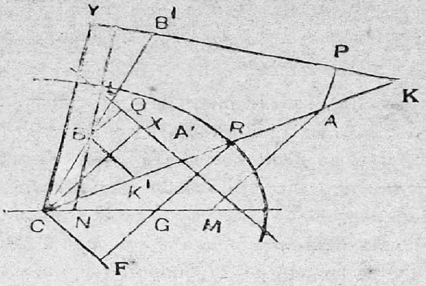
I have derived the above theorem analytically in order to establish with reference to Salmon's proof, its claim to be regarded as *the* extension of Salmon's theorem. But the result is in a sense a trivial modification of that used by Krishnaswami Aiyangar, for if p, q, r are the distances from the centre to the polars of A, B, C, the latter result gives the three equalities

$$\frac{B_a}{p} = \frac{A_b}{q}, \quad \frac{C_b}{q} = \frac{B_c}{r}, \quad \frac{A_c}{r} = \frac{C}{p},$$

which have only to be multiplied together. Conversely, if in our result we suppose C to tend to the centre, the ratio A_c/B_c tends, by the usual argument, to unity, and we have the equality directly related to Salmon's. Thus any geometrical proof of Krishnaswami Aiyangar's relation is in effect a geometrical proof of Theorem 1 above.

2. In view of the satisfying simplicity of this theorem, a little more may be said of the geometrical results with which it is connected. Let

me recall the construction in the April number of the *Journal*: the diameter through **A** cuts the polars of A and B in A′ and K, the



diameter through B cuts the same lines in L and B′, and the line through B parallel to the polar of A cuts the diameter through A in K′. By this construction CB/CL = CK′/CA′, if the diameters meet in C, and as soon as we recognise that BK′ is the polar of K, we have CK.CK′ = CA . CA′ and therefore CK′ / CA′ = CA / CK. For a parabola, BL = K′A′ = AK. The crucial step, that BK′ is the polar of K, may be put in the form of

THEOREM 2. *If B, B′ are conjugate points on one diameter of a conic and K, K′ are conjugate points on another diameter, then if the line BK′ is an ordinate to one of the diameters, the line B′K is an ordinate to the other.*

This result is immediately serviceable in certain elementary constructions. If we are given a pair of conjugate diameters p, d, of a conic, a point Q on the curve, and the tangent at Q, the conic is determined. In fact, if the tangent cuts p in T and d in t, and if the lines through Q parallel to d, p cut p, d in V, v, the equation of the conic is

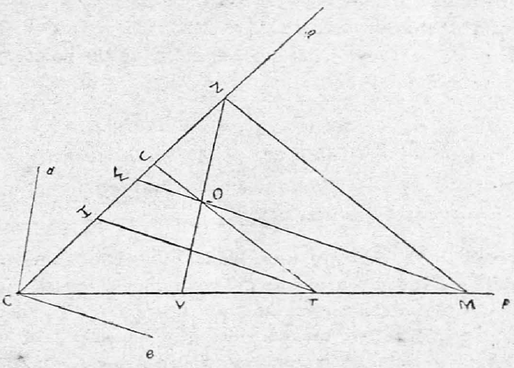
$$\frac{x^2}{\mathrm{CV}\,.\,\mathrm{CT}} + \frac{y^2}{\mathrm{C}v\,.\,\mathrm{C}t} = 1$$

If instead of the tangent at Q we are given a second point R, it is evident analytically that the conic is determined, for the equation of the

conic is simply

$$\begin{vmatrix} x^2, & y^2, & 1 \\ x_Q^{\,2}, & y_Q^{\,2}, & 1 \\ x_R^{\,2}, & y_R^{\,2}, & 1 \end{vmatrix} = 0.$$

To connect this second case geometrically with the first, we really require a construction for the tangent at Q. To be given the second point R is effectively to be given a second pair of conjugate diameters, namely, the diameter parallel to QR and the diameter which bisects QR. With a slight change of notation we may say therefore that we are given two pairs of conjugate diameters, p, d, and q, e, and a point O, and that we require to draw the tangent at O. Suppose the tangent, when drawn to cut p in T and q in U, let the line through O parallel to d cut p, q in V, N and let the line through O parallel to e cut p, q in M, W; also let the line through T parallel to e cut q in H. Because OV is an ordinate



to p and OT is the tangent at O, the points V, T are conjugate points on p. Since NV is an ordinate to p and HT is an ordinate to q, it follows that N, H are conjugate points on q, and since further W, U are conjugate points on q, CW. CU = CN.CH. Hence CU/CN = CH/CW = CT/CM, whence TU is parallel to MN, and we have :

THEOREM 3. *If p, q are any two diameters of a conic which are not conjugate to each other, and if the ordinates to q, p through a point O of the curve cut p, q in M, N, the tangent at O is parallel to MN.*

Instead of saying that the tangent at O is parallel to MN we may say that MN itself is an ordinate to the diameter CO. If we replace O by any other point of CO, and repeat the construction, we shall reach a line which is parallel to MN, that is, we shall reach another ordinate to CO. In other words, we have a construction for an ordinate to a given diameter, provided only that this diameter cuts the curve. The construction does not involve the actual points of intersection with the conic, and it is easy to shew that the existence of these points is not necessary.

If p, q are any two diameters of a conic which are not conjugate to each other, and if the ordinates to q, p through any point O of the plane cut p, q in M, N, then MN is an ordinate to the diameter through O.

We can express this result in simpler and more symmetrical terms:

THEOREM 4. *If three points X, Y, Z are so related to a conic that the diameter through X bisects chords parallel to YZ and the diameter through Y bisects chords parallel to ZX, then also the diameter through Z bisects chords parallel to XY.*

The theorems enunciated in this Note are all true for the parabola. For direct proofs some obvious modifications of argument are necessary, but such proofs are for the most part superfluous, even from the point of view of pure geometry, for once a result has been proved for a central conic the centre may be allowed to tend to infinity.

3. As remarked in the opening paragraph, the theorem on which we set out to comment is one of two results which combine to yield Sawayama's form of the extension of Salmon's theorem. The other of these results is, in the notation of the figure already referred to, that if the line through A perpendicular to its polar cuts an axis of the conic in M and the line through B perpendicular to its polar cuts the same axis in N, then MA . CX = NB . CY; in other words, since A and B are independent, the product MA . CX is an absolute constant. I cannot think that Krishnaswami Aiyangar is quite happy in his proof of this

step, since his method involves the use of a point in which CA is supposed to cut the conic. It is perhaps better to rely on more fundamental considerations. Let the polar of A cut the axis in U, and let the perpendicular from A on the axis be AV. Then from the right-angled triangles CXU, MYA we have

$$MA \cdot CX = MV \cdot CU = CV \cdot CU - CM \cdot CU.$$

But U, V are conjugate points on the axis, and the product CV . CU is constant. That the product CM . CU also is constant is a classical result, on which depends, for example, the theorem that is the basis of the theory of the normals from a point to a conic, the theorem, namely, that if the polar XU passes through a fixed point O, the envelope of AM is a parabola. The actual values of the two constants CV . CU and CM . CU are well known, but are not relevant to the present discussion.

E. H. NEVILLE.

Review

A TEXT-BOOK OF DYNAMICS, *with an Appendix on the Common Catenary, for the use of B.A. and B.Sc. Students of Indian Universities*, by K. D. PANDAY, M.A., B.Sc. (*Bajrang Press, Allahabad*).

This book has not been intended by the author, to give an account of the principles of Dynamics, but to help the student in mastering the mathematical technique necessary to solve Dynamical problems. In this respect, it fulfils its purpose, by its numerous collection of worked examples, carefully chosen so as to elucidate points at which difficulties of procedure and principle involved, most usually arise. It is a matter of ordinary experience to the teacher of mechanics, that the students never actually grasp any principle, till it is presented to them in relation to a concrete problem. The author has brought his own experience to bear, in the matter of exposition and the choice of the typical examples, and we have no doubt that it will be of great use to those who are undergoing the degree course.

R. V.

Solutions.

Question 256.

(T. SUNDARA RAO, B.A.) :—Show how to describe a cyclic quadrilateral whose sides, diagonals, radius of circumcircle and area are integers.

Solution by R. Ramachandra Rao.

Let $(x_1, y_1\ z_1)$ (x_2, y_2, z_2) and (x_3, y_3, z_3) be any three solutions integers of the equation

$$x^2 + y^2 = z^2$$

obtained, for example, by giving integral values to m and n in

$$m^2 - n^2,\ 2mn,\ m^2 + n^2.$$

Find the angles $\alpha_1, \alpha_2, \alpha_3$ given by

$$\sin\alpha_1 = \frac{x_1}{z_1},\quad \sin\alpha_2 = \frac{x_2}{z_2},\quad \sin\alpha_3 = \frac{x_3}{z_3}.$$

Then $\cos\alpha_1$, $\cos\alpha_2$, $\cos\alpha_3$ are also rational, being in fact

$$\frac{y_1}{z_1},\ \frac{y_2}{z_2},\ \frac{y_3}{z_3}.$$

Also

$\sin(\alpha_1 + \alpha_2)$, $\cos(\alpha_1 + \alpha_2)$, $\sin(\alpha_1 + \alpha_2 + \alpha_3)$, $\cos(\alpha_1 + \alpha_2 + \alpha_3)$

are likewise rational. Now imagine a cyclic quadrilateral inscribed in a circle of radius R, the angles subtended by the sides at the centre being

$$2\alpha_1,\ 2\alpha_2,\ 2\alpha_3 \text{ and } 2\alpha_4$$

where $$\alpha_4 = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3).$$

Then, its sides are $2R\sin\alpha_1$, $2R\sin\alpha_2$, $2R\sin\alpha_3$, $2R\sin\alpha_4$; its diagonals are $2R\sin(\alpha_1 + \alpha_2)$ and $2R\sin(\alpha_2 + \alpha_3)$ and its area is

$$\tfrac{1}{2}R^2(\sin 2\alpha_1 + \sin 2\alpha_2 + \sin 2\alpha_3 + \sin 2\alpha_4).$$

If R is rational, all these quantities are also rational. If R is chosen so as to be equal to $2z_1z_2z_3$ or a multiple of it, then it is clear that the sides, diagonals and area are all integers.

Also solved by S. Srinivasan.

[L. E. Dickson's *History of the Theory of Numbers*, (Vol. II, pp. 216—221) gives a full history of the problem of 'rational quadrilaterals' that is, quadrilaterals whose sides, diagonals and areas are rational. Brahmagupta and Bhascara are mentioned as the earliest investigators of this problem, of which the present question is only a special case.—ED.]

Question 1319.

(B. B. BAGI) :—ABC is a spherical triangle. Any point P on the sphere is joined to A, B, C and the cosines of these arcs are denoted by (a_1, b_1, c_1); similarly (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) denote the cosines of the arcs joining three other points Q, R, S to the vertices A, B, C. Prove that the points P, Q, R, S are on a small circle if

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ a_4 & b_4 & c_4 & 1 \end{vmatrix} = 0$$

Additional Solution by the Proposer.

A solution of the problem has already been published.* However the following method deserves to be known. Let l_r, m_r, n_r $(r = 1, 2, 3)$ be the direction cosines of A, B, C and λ_r, μ_r, ν_r $(r = 1, 2, 3, 4)$, those of P, Q, R, S, referred to rectangular axes through the centre of the sphere. Further take the radius of the sphere for the unit of length. It is then seen that $(\lambda_1, \mu_1, \nu_1)$ ... are the co-ordinates of P, Q, R, S.

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* *Vide J.I.M.S.*, XVI, (August 1925) p. 57.

Since these lie on a small circle, they must be co-planar.

$$\therefore \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 & 1 \\ \lambda_2 & \mu_2 & \nu_2 & 1 \\ \lambda_3 & \mu_3 & \nu_3 & 1 \\ \lambda_4 & \mu_4 & \nu_4 & 1 \end{vmatrix} = 0$$

Now multiplying this determinant by

$$\begin{vmatrix} l_1 & m_1 & n_1 & 0 \\ l_2 & m_2 & n_2 & 0 \\ l_3 & m_3 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

and observing that $\lambda_1 l_1 + \mu_1 m_1 + \nu_1 n_1 = a_1$, etc., we get the result.

Question 1436.

(B. B. BAGI):—A straight line OAB through a given point O meets two given planes P_1, P_2 in A and B. If $\{OO', AB\} = -1$, the point O' traces a plane (P_1P_2).

If there are three given planes $P_1P_2P_3$ then (P_2P_3), (P_3P_1), (P_1P_2) meet P_1, P_2, P_3 in straight lines lying in a plane $(P_1P_2P_3)$. If there are four planes $P_1P_2P_3P_4$ then the four straight lines in which $(P_2P_3P_4)$ etc. meet P_1 etc. lie in a plane $(P_1P_2P_3P_4)$ and so on.

Obtain a similar theorem in plane geometry.

By drawing all planes to pass through a given point and cutting them by a sphere with its centre at that point, obtain a theorem in spherical Trigonometry.

If the planes $P_1P_2P_3P_4$ form a tetrahedron whose pairs of opposite edges are at right angles and the point O is the *orthocentre*, prove that the plane $(P_1P_2P_3P_4)$ is perpendicular to the join of the centroid and the circumcentre of the tetrahedron.

Solution by M. V. Seshadri.

Let O be taken as the origin and let the equations to the planes P_1, P_2, P_3, P_4 etc. be

$$u_r \equiv a_r x + b_r y + c_r z + d_r = 0 \qquad (r = 1, 2, 3, 4 \ldots).$$

Then the equation to the plane through O and the line of intersection of $P_1 P_2$ is given by

$$u_1 + \lambda u_2 = 0 \quad \text{with} \quad d_1 + \lambda d_2 = 0.$$

i.e., by $$\frac{u_1}{d_1} - \frac{u_2}{d_2} = 0.$$

As O′ is the harmonic conjugate of O, the locus of O′ will be

$(P_1 P_2)$: $$\frac{u_1}{d_1} + \frac{u_2}{d_2} = 0.$$

Similarly we get $(P_2 P_3)$, $(P_3 P_1)$ etc.

Now $$\frac{u_1}{d_1} + \frac{u_2}{d_2} + \mu u_3 = 0$$

is a plane through the intersection of $(P_1 P_2)$ with P_3. If μ is put equal to $\frac{1}{d_3}$, it is easily seen to pass through the lines of intersection of $(P_2 P_3)$ with P_1 and $(P_3 P_1)$ with P_2. Thus $(P_1 P_2 P_3)$ has for its equation

$$\frac{u_1}{d_1} + \frac{u_2}{d_2} + \frac{u_3}{d_3} = 0.$$

Similarly by considering

$$\frac{u_1}{d_1} + \frac{u_2}{d_2} + \frac{u_3}{d_3} + \frac{u_4}{d_4} = 0,$$

the theorem is extended to four planes. This is the equation to

$$(P_1 P_2 P_3 P_4).$$

The argument can be extended to any number of planes.

The corresponding theorem in plane geometry is as follows:—

A straight line OAB through a given point O meets two lines $L_1 L_2$ in A and B. If (OO′, AB) = —1, then O′ traces a line $(L_1 L_2)$. Given three lines $L_1 L_2 L_3$ the points of intersection of $(L_1 L_2)$ with L_3, of $(L_1 L_3)$ with L_2, and of $(L_2 L_3)$ with L_1 lie on a line $(L_1 L_2 L_3)$. With four lines L_1, L_2, L_3, L_4, the four points of intersection of

$$(L_1 L_2 L_3),\ (L_2 L_3 L_4),\ (L_4 L_3 L_1) \text{ and } (L_4 L_1 L_2)$$

with L_4, L_1, L_2 and L_3 respectively lie on a line $(L_1 L_2 L_3 L_4)$ and so on. The equation to the line $(L_1 L_2 L_3 L_4 \ldots)$ with O as origin

would be

$$\sum \frac{l_r}{c_r} = 0$$

where c_r is the constant term and $l_r = 0$ the equation of the line L_r.

Again if the sphere be described to pass through O, the planes would intercept great circles on the sphere $G_1, G_2, G_3, \ldots$ If a great circle through O meets G_1, G_2 in A, B and O′ is taken so that

$$P\{OO', AB\} = -1,$$

P being the centre, the locus of O′ would be a great circle $(G_1\ G_2)$. For three great circles $G_1G_2G_3$, the points of intersection of (G_1G_2) with G_3, (G_2G_3) with G_1, and (G_3G_1) with G_2 lie on a great circle $(G_1G_2G_3)$, and so on.

To prove the second part, let O be taken as the origin and the axes be parallel to a pair of opposite edges V_1V_2, V_3V_4 and the shortest distance between V_1V_2 and V_3V_4. Let the equations to V_1V_2 and V_3V_4 be $x = 0, z = a$ and $y = 0, z = b$ respectively, and the co-ordinates of V_1, V_2, V_3 and V_4 be $(0, y_1, a)$, $(0, y_2, a)$, $(x_1, 0, b)$ and $(x_2, 0, b)$ respectively.

The equations to the planes $P_1\,P_2\,P_3\,P_4$ would be got by forming the equations to the four planes, through the points three at a time of $V_1V_2V_3V_4$.

The plane through $V_1V_2V_3$ has for equation

$$z - a + \lambda x = 0 \text{ with } b - a + \lambda x_1 = 0.$$

Calling this the plane P_1, we have

$$P_1 \equiv x(a-b) + (z-a)x_1 = 0.$$

Similarly
$$P_2 \equiv x(a-b) + (z-a)x_2 = 0$$
$$P_3 \equiv y(a-b) - (z-b)y_1 = 0$$
and
$$P_4 \equiv y(a-b) - (z-b)y_2 = 0$$

$\therefore$ $(P_1P_2P_3P_4)$ is the plane $\dfrac{u_1}{ax_1} + \dfrac{u_2}{ax_2} - \dfrac{u_3}{by_1} - \dfrac{u_4}{by_2} = 0.$

Now V_1V_3 and V_2V_4 are at right angles.

Their equations are

$$\frac{x - x_1}{x_1} = \frac{y}{-y_1} = \frac{z-b}{b-a}, \text{ and } \frac{x - x_2}{x_2} = \frac{y}{-y_2} = \frac{z-b}{b-a}$$

$$\therefore \qquad x_1 x_2 + y_1 y_2 + (a-b)^2 = 0$$

If the shortest distance be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$,

we have $\qquad lx_1 - my_1 + n(b-a) = 0,$

and $\qquad lx_2 - my_2 + n(b-a) = 0,$

giving $\qquad \frac{l}{(y_2 - y_1)(b-a)} = \frac{m}{(x_2 - x_1)(b-a)} = \frac{n}{x_2 y_1 - x_1 y_2}$

The shortest distance must be co-planar with $V_1 V_3$.

If the plane be $Ax + By + Cz = 0$,

$$\begin{aligned} Al + Bm + Cn &= 0 \\ By_1 + Ca &= 0 \\ Ax_1 \qquad + Cb &= 0, \end{aligned}$$

giving $\begin{vmatrix} l & m & n \\ 0 & y_1 & a \\ x_1 & 0 & b \end{vmatrix} = lby_1 + max_1 - nx_1y_1 = 0 \quad \ldots \quad (1)$

Similarly $\qquad lby_2 + max_2 - nx_2y_2 = 0 \quad \ldots \quad (2)$

Eliminating l between (1) and (2) and taking into account the ratio of m to n, we have, on simplifying $y_1 y_2 = a(b-a)$ ... (3)

Similarly eliminating m, between (1) and (2), and taking into account the ratio of l to n, we have $x_1 x_2 = -b(b-a)$ ... (4)

The circumsphere of the tetrahedron ABCD is given by the determinant

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + b^2 & x_1 & 0 & b & 1 \\ x_2^2 + b^2 & x_2 & 0 & b & 1 \\ y_1^2 + a^2 & 0 & y_1 & a & 1 \\ y_2^2 + a^2 & 0 & y_2 & a & 1 \end{vmatrix} = 0$$

Hence, if ξ, η, ζ be the co-ordinates of the circum-centre,

$$K\xi = \frac{1}{2}\begin{vmatrix} x_1^2 + b^2 & 0 & b & 1 \\ x_2^2 + b^2 & 0 & b & 1 \\ y_1^2 + a^2 & y_1 & a & 1 \\ y_2^2 + a^2 & y_2 & a & 1 \end{vmatrix} = \frac{1}{2}(x_1 - x_2)(x_1 + x_2)(y_1 - y_2)(a - b)$$

$$K\eta = -\frac{1}{2}\begin{vmatrix} x_1^2 + b^2 & x_1 & b & 1 \\ x_2^2 + b^2 & x_2 & b & 1 \\ y_1^2 + a^2 & 0 & a & 1 \\ y_2^2 + a^2 & 0 & a & 1 \end{vmatrix} = \frac{1}{2}(y_1^2 - y_2^2)(x_1 - x_2)(a - b)$$

and

$$K\zeta = \frac{1}{2}\begin{vmatrix} x_1^2 + b^2 & x_1 & 0 & 1 \\ x_2^2 + b^2 & x_2 & 0 & 1 \\ y_1^2 + a^2 & 0 & y_1 & 1 \\ y_2^2 + a^2 & 0 & y_2 & 1 \end{vmatrix} = \frac{1}{2}(x_1 - x_2)(y_1 - y_2)(x_1x_2 - y_1y_2 + a^2 - b^2)$$

$= \frac{1}{2}(x_1 - x_2)(y_1 - y_2)\, 2(a^2 - b^2)$ by (3) and (4).

where $$K = \begin{vmatrix} x_1 & 0 & b & 1 \\ x_2 & 0 & b & 1 \\ 0 & y_1 & a & 1 \\ 0 & y_2 & a & 1 \end{vmatrix} = (x_1 - x_2)(y_1 - y_2)(a - b)$$

Hence $\xi = \dfrac{x_1 + x_2}{2}$, $\eta = \dfrac{y_1 + y_2}{2}$, $\zeta = a + b$.

The co-ordinates ξ', η', ζ', of the centroid are

$$\frac{x_1 + x_2}{4},\quad \frac{y_1 + y_2}{4},\quad \frac{b + a}{2}.$$

$\therefore$ $\xi - \xi'$, $\eta - \eta'$, $\zeta - \zeta'$ are proportional to $x_1 + x_2$, $y_1 + y_2$ and $2(b + a)$ respectively. Now

$$(P_1 P_2 P_3 P_4) \equiv \frac{u_1}{ax_1} + \frac{u_2}{ax_2} - \frac{u_3}{by_1} - \frac{u_4}{by_2} = 0$$

when expanded becomes

$$\frac{x(a - b)(x_1 + x_2)}{ax_1 x_2} - \frac{y(a - b)(y_1 + y_2)}{by_1 y_2} + \frac{2z(b + a)}{ab} = 0$$

or $$x(x_1 + x_2) + y(y_1 + y_2) + 2z(a + b) = 0$$

which is perpendicular to the line with direction cosines proportional to

$$x_1 + x_2, \quad y_1 + y_2, \quad 2(a + b).$$

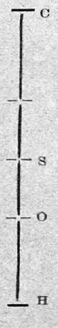
Hence $(P_1\ P_2\ P_3\ P_4)$ is perpendicular to the join of the circumcentre and centroid of the tetrahedron.

Question 1450.

(N. DURAIRAJAN) :—Show that if three consecutive tangents be drawn to a curve, the circumradius of the triangle so formed is ultimately $\frac{1}{4}$ of the radius of curvature of the curve at the point, and that the circumcentre and orthocentre of the triangle divide harmonically the line joining the point and its centre of curvature.

Solution by M. V. Seshadri and B. Ramamurthi.

If BA, BC, CA be consecutive tangents in order, in the limit, the circle of curvature is the ex-circle opposite A. So, ρ, the radius of curvature is the limiting value of $4R \sin A/2 \cos B/2 \cos C/2$, where A tends to 180° and B and C to zero.



Therefore $\rho = 4R$, R being the ultimate circum-radius of the triangle formed by the tangents.

Second part. Ultimately the orthocentre H, and the circum-centre S are on the normal at the point O considered on the curve. Let C be the centre of curvature. In a triangle, the distance of the vertex from the orthocentre is double the distance of the circum-centre from the opposite side and has the same sense. As here the vertex coincides with O, and the opposite side with the tangent at O, H is on SO produced so that $OH = 2SO$.

Now $$\frac{SO}{SC} = -\frac{1}{3} \text{ and } \frac{HO}{HC} = \frac{2OS}{6OS} = \frac{1}{3}.$$

Thus OC is divided harmonically at S and H.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1557. (R. VAIDYANATHASWAMY):—Let x and y be rational functions of position on a given Riemann surface, of the respective orders m, n (the *order* of a rational function on the Riemann surface, being the number of positions on the surface, at which the function takes a given value). Prove that x and y are connected by an algebraic relation of orders n, m in x, y respectively.

If z is a third rational function of order t on the surface, and if $f(x, z) = 0$, $\phi(y, z) = 0$ be the algebraic relations between x, z and y, z, shew that the eliminant of z between $f = 0$ and $\phi = 0$, is the product of t distinct polynomials of orders n, m in x, y.

1558. (R. VAIDYANATHASWAMY):—Analyse and expose the fallacy in the following:—

A line in space necessarily meets a plane in a point p. Only one condition has to be satisfied for a second point q on the line to lie in the plane. Hence it is only one condition on an arbitrary line, for it to lie in a given plane.

1559. (Prof. D. C. PAVATE):—Prove that, if $\phi(x)$ is differentiable in the *closed* range $(a \leqslant x \leqslant b)$ and $\phi(a) = \phi(b)$, then there exists at least one ξ in (a, b) such that

$$|\phi'(\xi)| + \frac{4\phi(a)}{b-a} > \frac{4}{(b-a)^2}\int_a^b \phi(x)\,dx.$$

1560. (B. RAMAMURTHI):—The normals at A, B, C to a circum-conic are concurrent at O. $A_1B_1C_1$ is the pedal triangle of O with respect to ABC; $A_2B_2C_2$ is the pedal triangle of O with respect to $A_1B_1C_1$ and so on. Show that there is a conic touching the sides of $A_nB_nC_n$ at A_{n+1}, B_{n+1}, C_{n+1}, the normals at which are concurrent at O.

1561. (S. D. CHOWLA):—If p is a prime $\equiv 9 \pmod{16}$, prove that

$$\sum_{1}^{\frac{1}{2}(p-1)} \left(\frac{n}{p}\right)\left(\frac{n^8+1}{p}\right) = 0$$

where (n/p) is the Legendre quadratic residue symbol.

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